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THE h-p VERSION OF THE BOUNDARY ELEMENT METHOD WITH GEOMETRIC MESH ON POLYGONAL DOMAINS

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The h-p Version of the Boundary Element Method with Geometric Mesh on Polygonal Domains

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Summary

This paper applies the techniques of the h-p version to the boundary element method for boundary value problems on plane non-smooth domains with piecewisely analytic boundary and data. The exponential rate of convergence of the boundary element Galerkin solution is obtained when a geometric mesh refinement is used near the vertices.

1. Introduction

The h, p and h-p versions (on a quasiuniform mesh) of the boundary element Galerkin method for integral equations on polygons has been studied in various papers, e.g., in [1, 2, 3, 4]. In all versions the algebraic rate of convergence of the Galerkin solution is restricted by the vertex singularities of the solution of the integral equations although it could be very smooth away from the corners. Based upon a regularity analysis (in countable normed spaces) for the solution of the integral equation we show in [5] that an exponential rate of convergence with respect to the number of degrees of freedom can be achieved for the h-p version by simultaneously reducing the mesh size and increasing the polynomial degrees of the boundary elements if a geometric mesh refinement towards the vertices is used. Here we report the main results from [5].

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with vertices A_i, $1 \leq i \leq M$, the boundary $\partial\Omega$ be a piecewise analytic curve

$$\partial\Omega = \Gamma = \bigcup_{i=1}^{M} \bar{\Gamma}_i$$

where Γ_i is an open line segment connecting A_i and A_{i+1} ($A_{M+1} = A_1$). By ω_i we denote the interior angle at A_i.

Let $H^k(\Omega)$, $k \geq 0$ integer, denote the usual Sobolev spaces furnished with the norm

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$$\|u\|_{H^k(\Omega)} = \Big(\sum_{0\le|\alpha|\le k} \|D^\alpha u\|^2_{L_2(\Omega)}\Big)^{1/2}$$

where $\alpha = (\alpha_1,\alpha_2)$, $\alpha_i \geq 0$ integer, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha u = \dfrac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. The space $H^{k-1/2}(\Gamma)$ is defined as the restriction of functions in $H^k(\Omega)$, i.e.,

$$H^{k-1/2}(\Gamma) = \{u|_\Gamma,\ u \in H^k(\Omega)\}, \quad \text{for } k > \tfrac{1}{2},$$

$$H^{k-1/2}(\Gamma) = L^2(\Gamma), \quad \text{for } k = \tfrac{1}{2},$$

$$H^{k-1/2}(\Gamma) = (H^{-(k-1/2)}(\Gamma))' \text{ (dual spaces)}, \quad \text{for } k < \tfrac{1}{2}.$$

For the investigation of singularities at corners we introduce weighted Sobolev spaces and countable normed spaces on the boundary Γ. Let $I = (a,b)$, and for $x \in (a,b)$, $r_1 = |x-a|$, $r_2 = |x-b|$, $\hat\Phi_{\hat\beta+k}(x) = \prod_{i=1}^{2} r_i^{\hat\beta_i+k}(x)$, $\hat\beta = (\hat\beta_1,\hat\beta_2)$, $0 < \hat\beta_1,\hat\beta_2 < 1$, k integer. Now we define for $k \geq \ell \geq 0$, and integer $\ell \geq 0$:

$$H^{k,\ell}_{\hat\beta}(I) = \{u \in H^{\ell-1}(I) \text{ if } \ell > 0;\ \|\hat\Phi_{\hat\beta+m-\ell}(x)u^{(m)}(x)\|_{L^2(I)} < \infty, \text{ for } \ell \leq m \leq k\}$$

$$B^{\ell}_{\hat\beta}(I) = \{u \in H^{k,\ell}_{\hat\beta}(I),\ \forall\ k \geq \ell,\ \|\hat\Phi_{\hat\beta+k-\ell}u^{(k)}(x)\| \leq Cd^{k-\ell}(k-\ell),\ C \geq 1,\ d \geq 1 \text{ independent of } k\}.$$

For any $\Gamma_i \subset \Gamma$, $H^{k,\ell}_{\hat\beta}(\Gamma_i)$ and $B^{\ell}_{\hat\beta_i}(\Gamma_i)$ are defined correspondingly with $\hat\beta_i = (\hat\beta_{i,1},\hat\beta_{i,2})$. $B^{\ell}_{\hat\beta}(\Gamma) = \prod_{i=1}^{M} B^{\ell_i}_{\hat\beta_i}(\Gamma_i)$ with $\ell = (\ell_1,\ell_2,\ldots,\ell_M)$, $\hat\beta = (\hat\beta_1,\ldots,\hat\beta_M)$. We shall write $\hat\beta_i \geq \tilde\beta_i$ (resp. $\hat\beta > \tilde\beta$) if $\hat\beta_{i,j} \geq \tilde\beta_{i,j}$, $j = 1,2$ (resp. $\hat\beta_i \geq \tilde\beta_i$, $1 \leq i \leq M$). By $B^{\ell,\ell+1}_{\hat\beta}(\Gamma)$ we denote for $1 \leq i \leq M$ the space $\prod_{0<\hat\beta_i<\frac{1}{2}} B^{\ell}_{\hat\beta_i}(\Gamma_i) \times \prod_{\frac{1}{2}<\hat\beta_i<1} B^{\ell+1}_{\hat\beta_i}(\Gamma_i)$.

Remark 1.1. $H^{k,\ell}_{\beta}(\Gamma)$ and $B^{k,\ell}_{\beta}(\Gamma)$ are the trace spaces of functions belonging to the weighted Sobolev space $H^{k,\ell}_{\beta}(\Omega)$ and the countable normed space $B^{k,\ell}_{\beta}(\Omega)$, respectively (see [6, 7, 8])

In the next sections we analyze the regularity of the solution of integral equations for Neumann and Dirichlet boundary value problems in terms of countable normed spaces $B^{\ell,\ell+1}_{\hat\beta}(\Gamma)$, design the geometric mesh and the distribution of the degrees of polynomials in the boundary element Galerkin method, which lead to the exponential rate of convergence with respect to the number of degree of

freedom. For the proof of the theorems we refer to [5].

2. The Neumann Boundary Value Problem

We consider the Neumann problem on a polygonal domain Ω

$$\begin{cases} u = 0 \text{ in } \Omega \\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma} = g \end{cases} \tag{2.1}$$

where $\frac{\partial u}{\partial n}$ means the normal derivative with respect to the outer unit normal, and g satisfies

$$\int_{\Gamma} g ds = 0 \tag{2.2}$$

Let D and K' be the integral operators for $x \in \Gamma$

$$Du(x) = -\frac{1}{\pi} \frac{\partial}{\partial n_x} \int_{\Gamma} \frac{\partial}{\partial n_y} (\ell n|x-y|) u(y) ds(y),$$

and

$$K' \frac{\partial u}{\partial n}(x) = -\frac{1}{\pi} \frac{\partial}{\partial n_x} \int_{\Gamma} \ell n|(x-y| \frac{\partial u(y)}{\partial n_y} ds(y)$$

Then the first kind integral equation for the Neumann boundary value problem (2.1) reads as

$$Du = f \text{ on } \Gamma \tag{2.3}$$

with $f = (1-K')g$, see [1].

<u>Theorem 2.1</u>. [5] *For given* $g \in \overset{0,1}{B_{\hat\beta}}(\Gamma)$ *satisfying* (2.2), *the integral equation* (2.3) *together with the side condition* $\int_{\Gamma} u ds = 0$ *has a unique solution* $u \in \overset{1,2}{B_{\tilde\beta}}(\Gamma)$, *for some* $\tilde\beta$, $0 < \tilde\beta < 1$ *where* $\tilde\beta$ *depends on* $\hat\beta$ *as well as on the geometry.*

Now we discuss the numerical solution of (2.3) by the h-p version of the boundary element Galerkin method with a geometric mesh.

Let Ω be a L-shape domain shown in Fig. 2.1. We assume for simplicity that the solution u of (2.3) belongs to $B^2_\beta(\Gamma)$ (resp. $B^1_\beta(\Gamma)$) with $\hat\Phi_{\hat\beta_1} = |y|^{\hat\beta_{1,2}}$, $\hat\Phi_{\hat\beta_2} = |x|^{\hat\beta_{2,1}}$, $\frac{5}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < 1$ (resp. $\frac{1}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < \frac{1}{2}$) and $\hat\Phi_{\hat\beta_j} = 1$ for $3 \le j \le 6$, i.e., the singularity occurs only at the origin. For example

this is the case of $u = r^{2/3} \cos \frac{2}{3}\theta$ (resp. $u = r^{1/3} \sin \frac{\theta}{3}$) on Γ where (r,θ) denote the polar coordinates centered at the origin (see [9, 10]).

Let $\sigma \in (0,1)$ be the mesh factor and n, integer, be the number of layers, and let $\Gamma_{i,j}$, $1 \le i \le I(j)$, $i \le j \le n+1$ be the boundary intervals such that $\mathrm{dist}(0,\Gamma_{i,j}) = \sigma^{n+1-j}$, $1 < j \le n+1$ and $\mathrm{dist}(0,\Gamma_{i,1}) = 0$, $1 \le i \le I(j)$. Then $\Gamma_\sigma^n = \{\Gamma_{i,j}, 1 \le i \le I(j), 1 \le j \le n+1\}$ is called the geometric mesh on Γ associated with σ and n. Fig. 2.2 shows a sequence of the geoemtric meshes with $\sigma = 0.15$.

Let $P = \{p_{i,j}, 1 \le i \le I(J), 1 \le j \le n+1\}$ be the degree vector with $p_{i,j} \ge 1$ integer. The boundary element space associated with the geometric mesh Γ_σ^n and degree vector P is defined by

$$S^P(\Gamma_\sigma^n) = \{\phi | \phi|_{\Gamma_{i,j}} \text{ is a polynomial of degree} \le p_{i,j}\}$$

and

$$\mathring{S}^P(\Gamma_\sigma^n) = S^P(\Gamma_\sigma^n) \cap C^0(\Gamma) \subset H^{1/2}(\Gamma)$$

The boundary element Galerkin procedure for the integral equation (2.3) with given $g \in B_{\beta}^{0,1}(\Gamma)$ is to find $u_p \in \mathring{S}^P(\Gamma_\sigma^n)$ satisfying $\int_\Gamma u_p ds = 0$ such that

$$\left\langle Du_p, w \right\rangle = \left\langle (1-K')g, w \right\rangle, \ \forall\ w \in \mathring{S}^P(\Gamma_\sigma^n) \tag{2.4}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. We have the following approximation result of the h-p version of the boundary element Galerkin method.

<u>Theorem 2.2</u>. [5] *Let* $u \in B_\beta^2(\Gamma)$ (*resp.* $B_\beta^1(\Gamma)$) *be the solution of the integral equation* (2.3) *and* Γ *be the boundary of the L-shaped domain shown in* Fig. 2.1, $\hat\beta_{i,j} = \hat\beta_{1,1} = \hat\beta_{2,2} = 0$, $3 \le i \le 6$, $j = 1,2$, $\frac{5}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < 1$, (resp. $\frac{1}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < \frac{1}{2}$) *and* Γ_σ^n, $\sigma \in (0,1)$ *be the geometric mesh on* Γ. *Let* $\mathring{S}^P(\Gamma_\sigma^n)$ *denote the boundary element space defined above with* $p_{ij} = p_j \ge 1$, $j\mu \le p_j \le \nu n$, $0 \le \mu \le \nu < \infty$. *Then the boundary element Galerkin solution* $u_p \in \mathring{S}^P(\Gamma_\sigma^n)$ *of* (2.4) *converges to* u *in* $H^{1/2}(\Gamma)$ *exponentially, i.e.,*

$$\|u - u_p\|_{H^{1/2}(\Gamma)} \le Ce^{-bN^{1/2}}$$

where N *is the number of degrees of freedom,* C *and* b *are some constants independent of* N.

3. The Dirichlet Boundary Value Problem

In this section we consider the Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_\Gamma = g \end{cases} \tag{3.1}$$

With the integral operators

$$V \frac{\partial u}{\partial n}(x) = -\frac{1}{\pi} \int_\Gamma \frac{\partial u(y)}{\partial n_y} \ell n|x-y| ds(y), \quad x \in \Omega$$

and

$$Ku(x) = -\frac{1}{\pi} \int_\Gamma \frac{\partial}{\partial n_y} \{\ell n|x-y|\} u(y) ds(y), \quad x \in \Omega$$

(3.1) leads, as shown in [2], to the first kind of integral equation

$$V \frac{\partial u}{\partial n} = f \tag{3.2}$$

with $f = (1+K)g$, for which there holds the following result

Theorem 3.1. [5] *Let* $\mathrm{cap}(\Gamma) \neq 1$ *where* $\mathrm{cap}(\Gamma)$ *is the analytic capacity of* Γ. *Then for given* $g \in B_\beta^{1,2}(\Gamma) \cap C^0(\Gamma)$ *there exists exactly one solution* $\frac{\partial u}{\partial n} \in B_{\tilde\beta}^{0,1}(\Gamma)$ *of* (3.2) *with some* $\tilde\beta$, $0 < \tilde\beta < 1$.

Now we consider the Galerkin solution of (3.2) obtained by the h-p version. For simplicity we assume again that Ω is the L-shaped domain shown in Fig. 2.1 and the solution $\frac{\partial u}{\partial n}$ of (3.2) has a singularity at the origin only. Then the geometric mesh Γ_σ^n on Γ and the boundary element space $S^P(\Gamma_\sigma^n)$ are defined as in the previous section. Obviously, $S^P(\Gamma_\sigma^n) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma)$.

The Galerkin procedure for the integral equation (3.2) is to seek $\psi_p \in S^P(\Gamma_\sigma^n)$ such that for all $w_p \in S^P(\Gamma_\sigma^n)$

$$\left\langle V\psi_p, \phi_p \right\rangle = \left\langle (1+K)g, \phi_p \right\rangle \tag{3.3}$$

For boundary element solutions ψ_p we have the following approximation theorem.

Theorem 3.2. [5] *Let* $\frac{\partial u}{\partial n} \in B_\beta^1(\Gamma)$ *(resp.* $B_\beta^0(\Gamma)$*) be the solution of the integral equation* (3.2) *and* Γ *is the boundary of the L-shaped domain shown in* Fig. 2.1, *with* $\mathrm{cap}(\Gamma) \neq 1$, *and* $\hat\beta_{1,1} = \hat\beta_{2,2} = \hat\beta_{i,j}$ *for* $3 \leq i \leq 6$, $j = 1,2$, $\frac{5}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < 1$, *(resp.* $\frac{1}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < \frac{1}{2}$*) and let* Γ_σ^n, $\sigma \in (0,1)$ *be the*

geometric mesh on Γ *and* $S^P(\Gamma_\sigma^n)$ *be the boundary element space defined above with* $p_{i,j} = p_j \geq 0$, $j\mu \leq p_j \leq \nu n$ $0 \leq \mu \leq \nu < \infty$. *Then the boundary element Galerkin solution* $v_p \in S^P(\Gamma_\sigma^n)$ *of* (3.3) *converges to* $\frac{\partial u}{\partial n}$ *in* $H^{-1/2}(\Gamma)$ *exponentially, i.e.,*

$$\|v - \frac{\partial u}{\partial n}\|_{H^{-1/2}(\Gamma)} \leq Ce^{-bN^{1/2}}$$

where N *is the number of degrees of freedom,* C *and* b *are some constants independent of* N.

4. Conclusion

The regularity results for the solutions of the integral equations for the Dirichlet and Neumann boundary value problems of the Laplacian can be generalized to bipotential and elasticity problems with essential, natural, and mixed boundary conditions. The h-p version of the boundary element method possesses advantages over the finite element method such as reducing the number of degree of freedom and avoiding the difficulties in the treatment of non-homogeneous essential boundary conditions in the finite element method (see [10]). Although the geometric mesh shown in Fig. 2.1 is designed for the problems with singularity at one corner it can be generalized without any difficulty to the case that the singularity occurs at several corners of Γ, and the exponential rate of convergence can be proven as well. All theorems above will hold if Ω is a curvilinear polygon.

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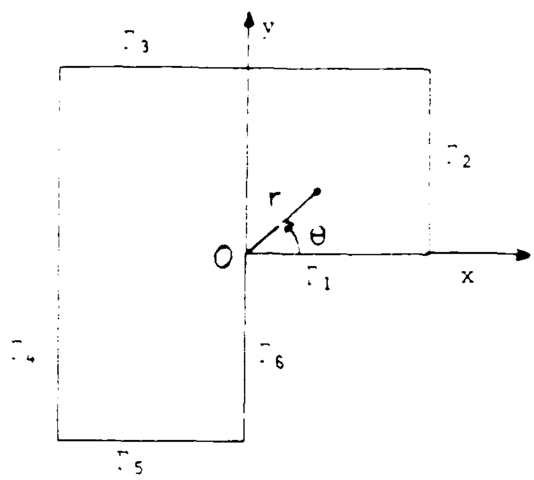

Figure 2.1. L-shaped Domain Ω

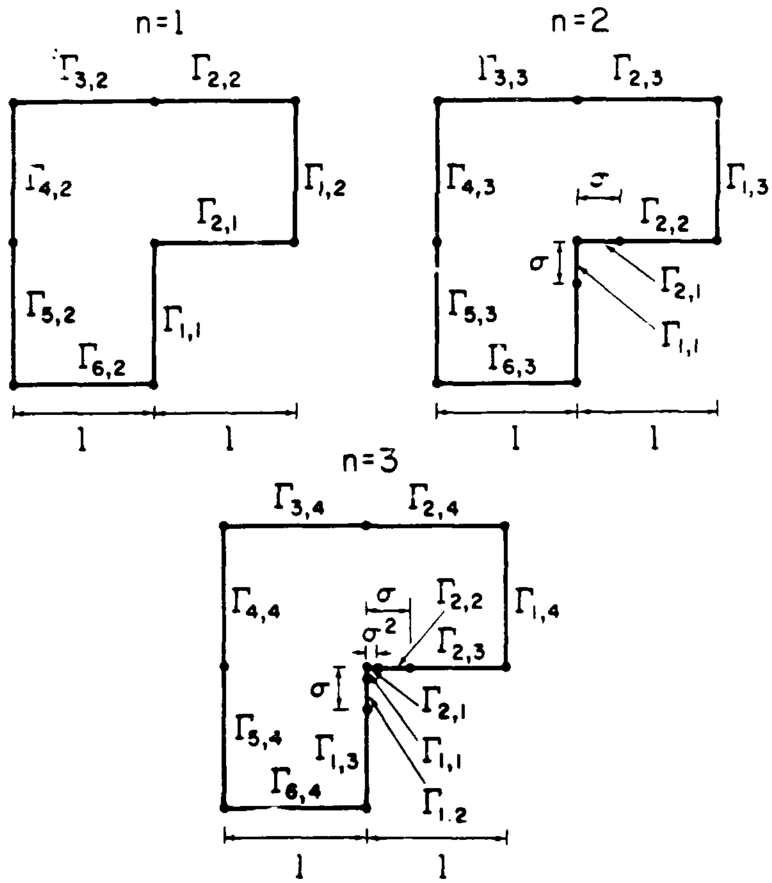

Figure 2.2. Geometric Mesh Γ_σ^n, n = 1,2,3, σ = 0.15.

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